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DYNAMIC RESPONSE OF ELASTIC–PLASTIC PIN-ENDED BEAMS BY GALERKIN'S METHOD

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Abstract—The problem of the elastic–plastic response of beams with pinned ends under short pulse loading is solved by Galerkin's method (with six degrees of freedom). Numerical results are compared with results obtained by the ABACUS program. An approximate "elastic recovery" type solution is proposed. It is shown that this solution gives satisfactory estimates for the lower bound of the long-term vibrations. Chaotic behavior of the beam is discussed; the achieved results are compared with the FEM solutions and with the solutions obtained by Symonds *et al.* (1986, *Int. J. Impact Engng* **4**, 72–82; 1991, *Int. J. Solids Structures* **27**, 299–314) for Shanley type models.

1. INTRODUCTION

In this paper, we treat the problem of elastic–plastic response of a fixed-ended beam subjected to a short intensive pulse of transverse loading. Interest in this problem was increased significantly in 1985 when Symonds and his collaborators described the phenomenon of "counter-intuitive" or "anomalous behavior" of the beam's response. According to this concept the permanent deflections of the beam may be in the opposite direction to the acting load. After 1985, Symonds and his coworkers published many papers about this problem [see e.g. Symonds and Yu (1985); Symonds *et al.* (1986), (1991); Symonds and Lee (1989); Lee and Symonds (1992); Lee *et al.* (1992)].

The solution of this problem requires a lot of computation time, and therefore there are only a few results obtained with the aid of ABACUS or other FEM-type programs [see Symonds and Yu (1985), Symonds *et al.* (1986), Symonds and Lee (1989)]. This circumstance forces one to seek simplified solutions. Symonds and his coworkers for the most part of their papers have used a Shanley type model, for which rigid bars are connected by one or two elastic–plastic cells. They have also shown that for a 2DoF (two degrees of freedom) model chaotic motion of the beam can take place. A different SDoF (single degree of freedom) model was proposed by Yu and Xu (1989): this consists of a rigid concentrated mass and two tiers of elastic–plastic springs, which connect the mass to the fixed pin ends. In the present paper the problem of dynamic response of a uniform beam is solved by Galerkin's method. Numerical results are compared with the results obtained by the ABACUS programs and also with the diagrams of Symonds *et al.* for a Shanley type model.

Lee and Symonds (1991) and Symonds *et al.* (1991) proposed an approximate so-called "elastic recovery model", according to which the beam's response subsequent to the peak deflection is wholly elastic. An analogical model was recommended by Lepik (1994a,b) and in these works a beam with fully fixed ends was considered. Now the same results will be carried over to a beam with pinned ends.

2. EQUATIONS OF MOTION

Let us consider an elastic–plastic beam with rectangular cross-section; b, h and L are the width, thickness and length of the beam, respectively. The beam is attached to smooth fixed pins and subjected to a pulse of uniform pressure, which is applied in the central part

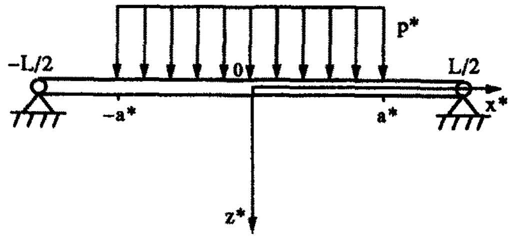

Fig. 1. Beam dimensions.

of the beam (Fig. 1). The axes of coordinates x^* and z^* are chosen according to Fig. 1. We shall introduce the following notations: t^*—time, u^*—axial displacement, w^*—deflection, ρ—density, E—Young's modulus, σ_s—yield stress, p^*—load intensity, $2a^*$—length of the loading region, T^*—axial force, M^*—bending moment.

The equations of motion are

$$\frac{\partial T^*}{\partial x^*} = \rho hb\frac{\partial^2 u^*}{\partial t^{*2}}, \quad \frac{\partial^2 M^*}{\partial x^{*2}} + \frac{\partial}{\partial x^*}\left(T^*\frac{\partial w^*}{\partial x^*}\right) + p^*(x^*) = \rho bh\frac{\partial^2 w^*}{\partial t^{*2}}. \tag{1}$$

It is convenient to pass to the dimensionless quantities:

$$x = \frac{x^*}{L}, \quad c = \sqrt{\frac{\sigma_s}{\rho}}, \quad t = \frac{c}{L}t^*, \quad u = \frac{u^*}{L}, \quad w = \frac{w^*}{h},$$

$$z = \frac{z^*}{h}, \quad p = \frac{p^*L^2}{\sigma_s bh^2}, \quad a = \frac{a^*}{L}, \quad T = \frac{T^*}{\sigma_s bh}, \quad M = \frac{4M^*}{\sigma_s bh^2}. \tag{2}$$

Equations (1) now get the form

$$T' = \ddot{u}, \quad M'' + 4(Tw')' + 4p(x) = 4\ddot{w} \tag{3}$$

where $p(x) = p$ for $|x| < a$ and $p(x) = 0$ for $|x| > a$. Henceforth, primes and dots denote differentiation with respect to x and t. We shall find the quantities T and M from the equations

$$T(x, t) = \int_{-0.5}^{0.5} \sigma(x, z, t)\,dz, \quad M(x, t) = 4\int_{-0.5}^{0.5} \sigma(x, z, t)z\,dz, \tag{4}$$

where $\sigma = \sigma^*/\sigma_s$ is the nondimensional stress.

We shall assume that the hypotheses of Kirchhoff hold, then the nondimensional axial deformation is

$$e = u' + \left(\frac{h}{L}\right)^2(\tfrac{1}{2}w'^2 - zw''). \tag{5}$$

3. METHOD OF SOLUTION

Equations (3) shall be integrated by the Galerkin method (due to symmetry we shall consider only one half of the beam $x \in [0, 0.5]$):

$$\int_0^{0.5}(T'-\ddot{u})\delta u\,\mathrm{d}x = 0\,,\quad \int_0^{0.5}[M''+4(Tw')'+4p(x)-4\ddot{w}]\delta w\,\mathrm{d}x = 0.$$

Integrating these equations by parts and taking into account boundary conditions $u = w = M = 0$ for $x = 0.5$ and the conditions $u = w' = M' = 0$ for $x = 0$ we get

$$\int_0^{0.5}(T\delta u'+\ddot{u}\delta u)\,\mathrm{d}x = 0,\quad \int_0^{0.5}(M\delta w''-4Tw'\delta w'+4p(x)\delta w-4\ddot{w}\delta w)\,\mathrm{d}x = 0. \tag{6}$$

Displacements u and w shall be sought in the form

$$u = A\sin 2\pi x+B\sin 4\pi x+C\sin 6\pi x \tag{7}$$
$$w = f\cos\pi x+g\cos 3\pi x+h\cos 5\pi x.$$

Now all boundary and symmetry conditions are satisfied. It follows from eqns (6), that

$$\ddot{A} = -8\pi J_1\,;\quad \ddot{B} = -16\pi J_2;\quad \ddot{C} = -24\pi J_3,$$

$$\ddot{f} = -4\pi^2(L_{11}f+3L_{12}g+5L_{13}h)-\pi^2K_1+\frac{4p}{\pi}\sin\pi a$$

$$\ddot{g} = -12\pi^2(L_{12}f+3L_{22}g+5L_{23}h)-9\pi^2K_2+\frac{4p}{3\pi}\sin 3\pi a$$

$$\ddot{h} = -20\pi^2(L_{13}f+3L_{23}g+5L_{33}h)-25\pi^2K_3+\frac{4p}{5\pi}\sin 5\pi a \tag{8}$$

where

$$J_i = \int_0^{0.5}T\cos(2i\pi x)\,\mathrm{d}x,\quad K_i = \int_0^{0.5}M\cos[(2i-1)\pi x]\,\mathrm{d}x$$

$$L_{ij} = \int_0^{0.5}T\sin[(2i-1)\pi x]\cdot\sin[(2j-1)\pi x]\,\mathrm{d}x$$

$$L_{ij} = L_{ji},\quad i,j = 1,2,3. \tag{9}$$

In order to calculate these integrals we must evaluate axial forces $T(x)$ and bending moments $M(x)$. This can be done in the following manner.

We shall confine our investigation to the ideal–plastic material (without strain-hardening). Since we must also take into account unloading, the stress–strain diagram has the form shown in Fig. 2. The segment FA corresponds to pure elastic deformations. For

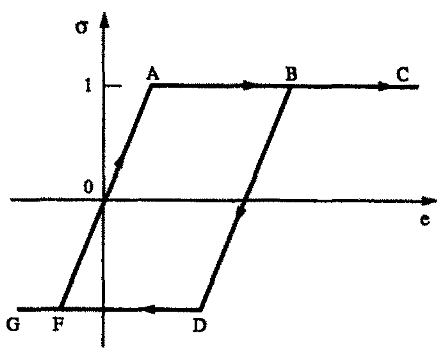

Fig. 2. Stress–strain diagram.

segments AC and DG we have plastic loading (with tensile stresses for $\sigma = 1$ and compression for $\sigma = -1$). In the case of the segment BD elastic unloading takes place.

We must know the values of σ and e from which unloading begins; we shall note these by symbols σ_m and e_m. For the instant $t = 0$ we shall take $\sigma_m = e_m = 0$. If plastic deformations develop in the beam, the values σ_m, e_m will change and we get the following cases.

(1) If $e_m - 2\sigma_s/E < e < e_m$, elastic unloading takes place and we have

$$\sigma = \sigma_m + \frac{E}{\sigma_s}(e - e_m). \tag{10}$$

(2) If $e > e_m$ and $\sigma_m < 1$ we also have elastic deformations; eqn (10) holds with the only exception, that if the value of σ calculated from (10) is greater than one, we must take $\sigma = \sigma_m = 1$.

(3) The case $e < e_m$ and $\sigma_m > -1$ is analogical to case (2); if eqn (10) gives a value $\sigma < -1$ we have to substitute $\sigma = \sigma_m = -1$.

(4) If $e > e_m$ and $\sigma_m = 1$ then we have plastic loading (with tensile stress) and we shall take $e_m = e$, $\sigma = 1$.

(5) If $e < e_m$ and $\sigma_m = -1$ we have plastic compression and must take $e_m = e$, $\sigma = -1$.

Knowing the function $\sigma = \sigma(x, z, t)$ for some instant t, we can evaluate the integrals (4) for this instant. After that we shall calculate the integrals (9) and the quantities $\ddot{A}$, $\ddot{B}$, $\ddot{C}$, $\ddot{f}$, $\ddot{g}$, $\ddot{h}$ in eqns (8). The coefficients A, B, C, f, g, h and their rates can be found according to the method of central differences.

In order to find out the applicability and efficiency of the proposed algorithm, we have made a comparison with the results computed by the ABACUS program [see Symonds and Yu (1985) and Symonds *et al.* (1986)]. The material and beam parameters are as follows $L = 200$ mm, $b = 20$ mm, $h = 4$ mm, $E = 80 \times 10^9$ N m^{-2}, $\sigma_s = 0.3 \times 10^9$ N m^{-2}, $\rho = 2700$ kg m^{-3}. The beam is subjected to the uniformly distributed load $p^* = 19.2$ N mm^{-1}, which acts up to time $t_1^* = 0.5$ ms and after that is removed. Our nondimensional quantities correspond to these data

$$\sigma_s/E = 0.00375, \quad h/L = 0.02, \quad p = 8, \quad t_1 = 0.8333.$$

By integrating eqns (8) we can evaluate the midpoint deflection $w(0)$ versus time t; this function is presented in Fig. 3 as curve 1. Curves 2–3 are taken from the papers quoted above, they demonstrate solutions obtained by ABACUS programs: curve 2 corresponds to a Timoshenko beam with the element mass lumped at nodes, curve 3, to a Euler–Bernoulli

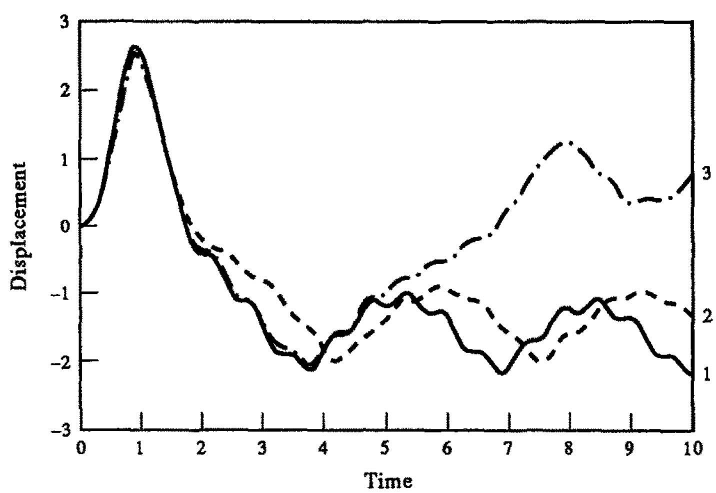

Fig. 3. Midpoint displacement–time history, 1—present solution; 2, 3—solutions according to ABACUS codes.

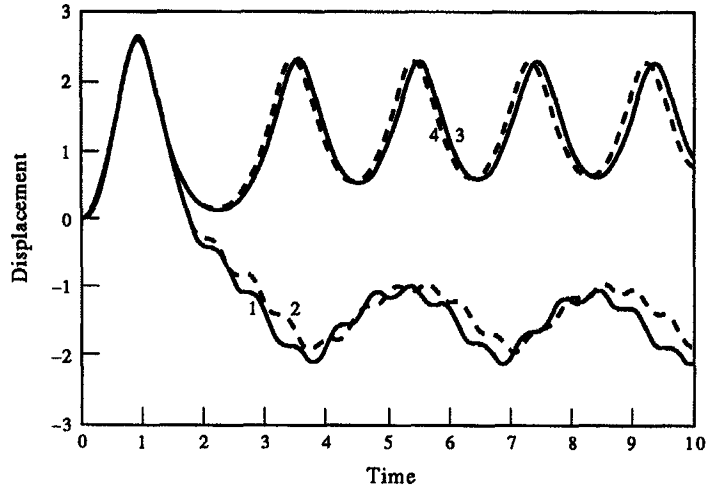

Fig. 4. Sensitivity of the beam's response to the number of coefficients in eqn (7); 1—6DoF solution, 2—solution for $C = h \equiv 0$, 3—solution for $B = C = g = h \equiv 0$, 4—solution for $g = h \equiv 0$.

beam with consistent mass matrix. Taking into account large discrepancies between different FEM codes [cf. Fig. 3 in the paper by Symonds and Yu (1985)], our solution can be regarded as quite reliable; the computing time for this is considerably less than in the case of the FEM codes.

It is interesting to clear up how sensitive the response history of the beam is to the number of coefficients in eqn (7). The results of some calculations, which were carried out for the same material and beam parameters as before, are presented in Fig. 4. It follows from this figure that curves 1 and 2 are very near each other; consequently exactness obtained by the 6DoF model may be regarded as wholly satisfactory. On the other hand, the 2DoF model (curve 3) does not describe properly the beam's response (in this case negative deflections do not take place at all). This picture does not alter if we take for **u** three unknown coefficients A, B and C (curve 4). It should be mentioned that such a discrepancy between solutions with different DoF takes place only in the region of anomalous behavior of the beam; outside this region the 2DoF solution $B = C = g = h = 0$ can give wholly applicable results.

4. SOLUTION WITH ELASTIC RECOVERY

The solution of the problem discussed above can be simplified to a great extent, if we assume that the motion subsequent to the peak deflection is entirely elastic. Such a solution was proposed by Symonds *et al.* [see Lee and Symonds (1991) and Symonds *et al.* (1991)]; they called it "the elastic recovery case". Although such a solution is artificial, it nevertheless describes the general features of the beam's response.

An analogical solution was given by Lepik (1994a) for a beam with fully clamped ends under transverse impulsive loading; its validity was discussed in Lepik (1994b) and it was shown that this method gives a simple method for evaluating minimal bounds for deflections.

In the present paper this method is applied for a beam with smooth pinned ends. We shall start from the following assumptions:

(i) the motion subsequent to the peak deflection is wholly elastic and proceeds inertially (i.e. $p = 0$ for this stage of motion);

(ii) plastic deformations dominate in such a way that at the instant when peak deflection is reached, the beam is practically in the membrane state;

(iii) we shall use the single degree of freedom model for which $A = B = C = 0, f \neq 0$, $g = h = 0$. Since the motion subsequent to the peak deflection is elastic, the quantities σ_m, e_m in eqn (10) can be treated as the values of σ and e at the peak deflection.

In the membrane state we have

$$T_m = \int_{-0.5}^{0.5} \sigma_m \, dz = 1, \quad M_m = 4\int_{-0.5}^{0.5} \sigma_m z \, dz = 0.$$

Further on we shall assume that $u = 0$, $w = f\cos\pi x$. Making use of eqn (10) and calculating the integrals (4), we obtain

$$T = 1+\tfrac{1}{2}\kappa\pi^2(f^2-f_m^2)\sin^2\pi x, \quad M = \tfrac{1}{3}\kappa\pi^2(f-f_m)\cos\pi x, \quad \left(\kappa = \frac{E}{\sigma_s}\frac{h^2}{L^2}\right). \tag{11}$$

In these formulas f_m stands for the value of f at peak deflection.

Now it follows from eqns (8)–(9) that

$$\ddot{f} = -\pi^2\left[f+\frac{\pi^2}{12}\kappa(f-f_m)+\frac{3}{8}\pi^2 f(f^2-f_m^2)\right]. \tag{12}$$

If the parameters f_m, κ are specified, eqn (12) can be integrated numerically. The first integral is

$$\tfrac{1}{2}\dot{f}^2 = -U, \tag{13}$$

where

$$U = \pi^2(f-f_m)\left[\frac{1}{2}(f+f_m)+\frac{\pi^2\kappa}{24}(f+f_m)+\frac{3\pi^2\kappa}{32}(f^2-f_m^2)(f+f_m)\right]$$

is the nondimensional potential energy (the constant in the energy expression has been chosen so that $U = 0$ for $f = f_m$).

Symonds and Lee (1989) have shown that for a SDoF beam, anomalous response of the beam is possible only if the energy curve $U = U(f)$ has two minima. Let us examine when this takes place.

First let us introduce a new variable $y = f-f_m$, the expression for the potential energy now takes the form

$$U = \pi^2 y\left[f_m+\left(\frac{1}{2}+\frac{\pi^2\kappa}{24}\right)y+\frac{3\pi^2\kappa}{32}y(y+2f_m)^2\right]. \tag{14}$$

According to eqn (13) the motion is possible only if $U < 0$. If f_m is small enough, the curve $U = U(y)$ has only one minimum. By increasing the quantity f_m a point of inflection appears, after that two minima are possible. At the inflection point we have $U''(y) = 0$ and

$$y = -f_m \pm \sqrt{\frac{1}{3}f_m^2-\frac{8}{9\kappa\pi^2}-\frac{2}{27}}.$$

The smallest value of f_m for which the inflection point appears is

$$f_m^- = \sqrt{\frac{2}{3}\left(\frac{1}{3}+\frac{4}{\pi^2\kappa}\right)}. \tag{15}$$

Counter-intuitive behavior is possible up to a value f_m^+ for which $U(y) = U'(y) = 0$ [if $f_m > f_m^+$ then the minima of $U = U(y)$ are separated by a region where $U > 0$]. This means that the polynom $U = U(y)$ must have a double root at $f_m = f_m^+$ and

$$U = \frac{3\pi^4\kappa}{32} y(y-y_1)^2(y-y_3) \tag{16}$$

where y_1 and y_3 are the roots of the equation $U(y) = 0$.

By comparing (14) and (16) we find

$$2y_1+y_3 = -4f_m^+, \quad y_1^2+2y_1y_3 = 4f_m^{+2}+\frac{16}{3\pi^2\kappa}+\frac{4}{9}, \quad y_1^2y_3 = -\frac{32f_m^+}{3\kappa\pi^2}. \tag{17}$$

By eliminating the quantities f_m and y_3 we get the biquadratic equation

$$y_1^4-\frac{16}{3\pi^2\kappa}\left(1+\frac{3}{\pi^2\kappa}\right)y_1^2+\frac{64}{9\pi^4\kappa^2}\left((1+\frac{12}{\pi^2\kappa}\right) = 0. \tag{18}$$

After solving this equation we find from (17) that

$$f_m^+ = \frac{2y_1^3}{\alpha-4y_1^2}, \quad y_3 = \frac{2y_1\alpha}{4y_1^2-\alpha}, \tag{19}$$

where $\alpha = 32/(3\pi^2\kappa)$.

Counter-intuitive behavior of the beam is possible in the region $f_m \in (f_m^-, f_m^+)$. Hereby it should be noted that the evaluation of these bounds is very simple and for this purpose we do not need to integrate eqn (12).

As a sample we shall take the beam considered by Lee *et al.* (1992). The beam and material parameters are as follows: $\sigma_s = 0.3$ GPa, $E = 40$ GPa, $\rho = 2700\ \mathrm{kg\,m^{-3}}$, $b = 0.0295$ m, $L = 0.2$ m, $h = 0.00271$ m. The load P^* has a fixed duration $t_1^* = 0.5\ 10^{-3}$ s. Passing over to our nondimensional quantities (2) we get $\kappa = 0.0245$, $t_1 = 0.833$. By integrating eqn (12) the lower bound of deflections has been found and presented in Fig. 5. Calculating the values of $f_m^\pm$ from eqns (15) and (18)–(19) we get $f_m^- = 3.35$ and $f_m^+ = 4.65$. Most important of these is the last quantity, since at $f_m = f_m^+$ the diagram has an abrupt change (we shall call this value critical, since for $f_m > f_m^+$ the lower bound of deflections is always positive).

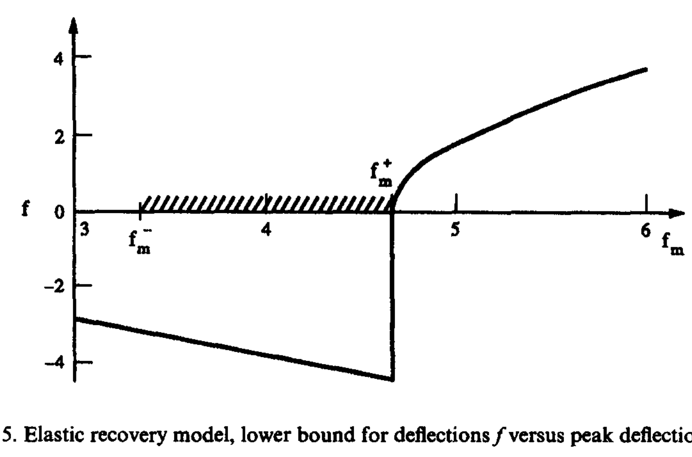

Fig. 5. Elastic recovery model, lower bound for deflections f versus peak deflections f_m.

For calculating f_m as a function of nondimensional load intensity p we have to integrate eqn (8) up to the instant f_m, corresponding to the peak deflection.

Comparison of the diagram in Fig. 5 with other solutions will be carried out in the next section.

5. DISCUSSION OF NUMERICAL RESULTS

Some computations were carried out to illustrate the method of solution presented in Sections 2–3. For the beam and material parameters again the values

$$\sigma_s/E = 0.0075, \quad h/L = 0.01355, \quad \kappa = 0.0245, \quad t_1 = 0.8333.$$

have been taken.

For the time step the value $\Delta t = 0.0005$ was chosen; the integrals (9) were evaluated by the Simpson's formula, the interval $x \in (0, 0.5)$ was divided into 10 parts (control computations showed that these values provide sufficient exactness of the achieved results).

If we evaluate the function $w = w(x)$ according to the second formula of (7) then the extremum of this function may not be in the midpoint $x = 0$. This is demonstrated in Fig. 6, where the function $w = w(x)$ is plotted for two instants [here calculations have been carried out for $p = 12$, the chosen instants correspond to cases where $\dot{w}(0) = 0$]. It follows from this figure that the midpoint deflection may not be a truthful criterion for characterizing the vibrations; therefore in the following instead of using it we shall use the mean deflection

$$W(t) = \int_0^{0.5} w(x, t)\,dx. \tag{20}$$

First we shall consider the case where the load is uniformly distributed along the beam. Some results, which are obtained by integrating eqns (8), are presented in Fig. 7. Curve 1 shows peak deflections versus load p; the curves 2–3 give an envelope of the mean deflections for the long-term motion. This envelope has a wide slot located approximately in the interval $15.3 < p < 17.1$ and near $p \approx 12.9$ a narrow region of irregularly positioned slots, where the maximal mean deflection lies on the negative side. This picture is quite similar to the solutions obtained by FEM techniques [see e.g. Fig. 1 from the paper by Lee *et al.* (1992)].

In order to clarify the character of the beam vibrations (especially near the slots) diagrams in Figs 8–10 are put together. In these diagrams, mean deflection histories, phase portraits and power spectra for four values of loads are presented. It follows from Figs 8–10 that we have irregular motion in the initial phase with transition to a smaller amplitude nonchaotic vibration (in this sense we can speak about transient or initial chaos). In Fig. 10, where spectral power R versus nondimensional frequency ω is presented, we see that for $p = 12.7$ and $p = 12.8$ the band of frequencies is broader than in the other two cases, implying that the chaotic effects in these cases are more essential.

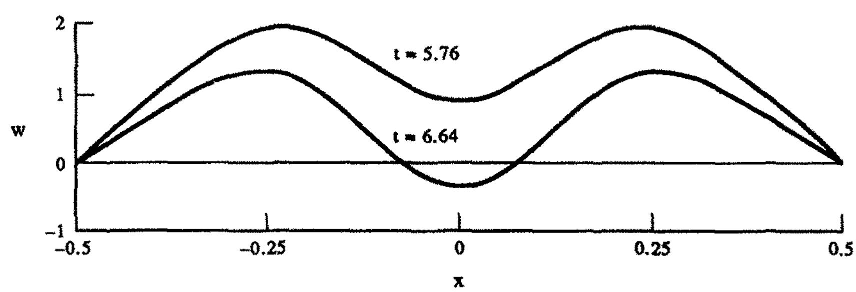

Fig. 6. Deflections along the beam for two instants.

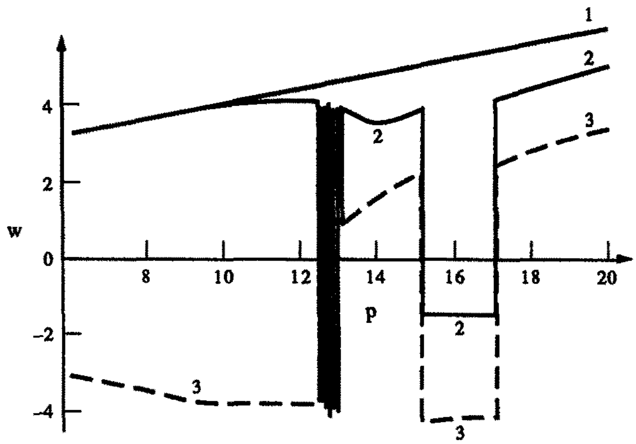

Fig. 7. Mean deflection versus load; 1—peak deflections, 2–3—envelope of minimal and maximal deflections [obtained by integrating the system (8)–(9)].

Analogical results may be obtained for a concentrated load, which is acting in the midpoint of the beam. For briefness, only some diagrams of deflection histories are presented here (Fig. 11). The values of the loads p^* have been chosen in accordance with the paper by Lee and Symonds (1992), except the value $P^* = 3000$ N.

Now we shall compare the diagram from Fig. 7 with the elastic recovery type solution (Fig. 5). The latter solution as an elastic SDoF solution cannot predict the slots, but outside that region it gives a rather good approximation for the minimal deflections. It follows also from Fig. 7 that dependence of the load p upon the mean peak deflection W_m is practically linear and can be approximated in the form

$$p = 5.05W_m - 10.62.$$

By taking for W_m the values of $f_m^{\pm}$, we get $p^- = 6.3$ and $p^+ = 12.9$ [the interval of load $p \in (p^-, p^+)$ determines the region in which according to the elastic recovery solutions

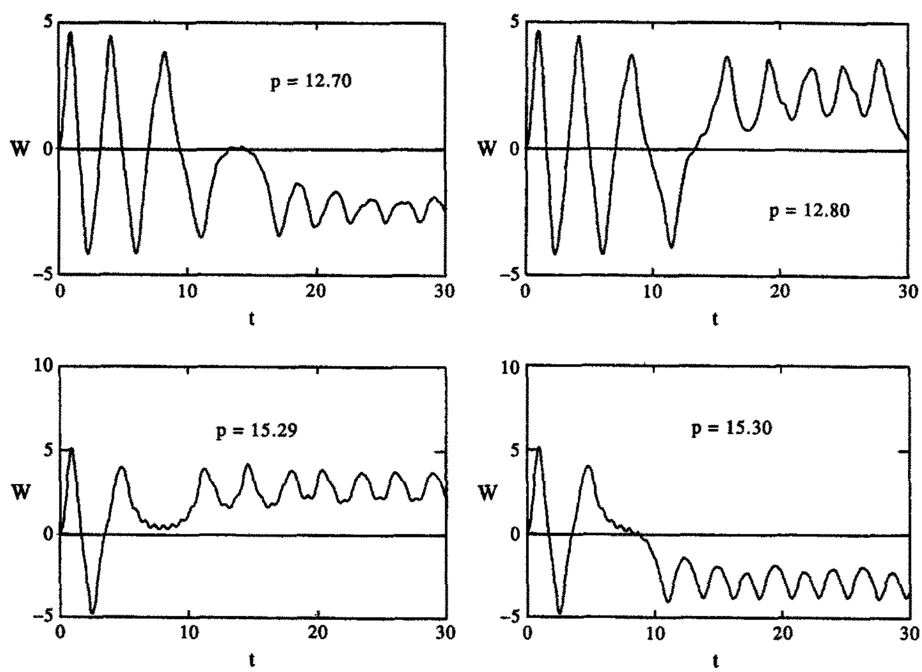

Fig. 8. Mean deflection histories of a beam under uniform transverse loading.

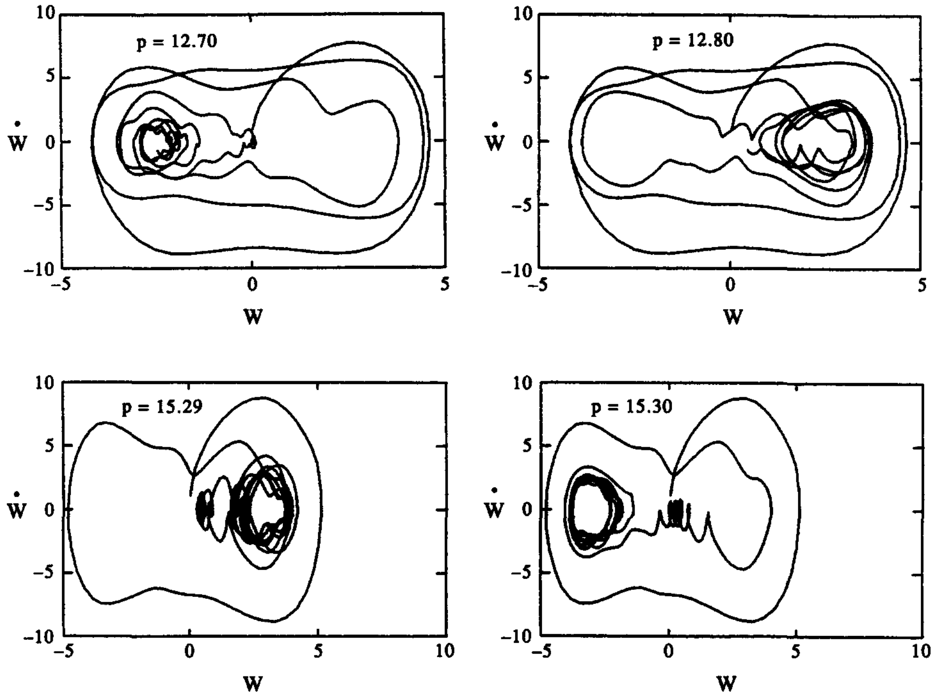

Fig. 9. Phase portraits of a beam under uniform transverse loading.

counter-intuitive behavior of the beam can take place]. Here it should be pointed out that the upper bound p^+ of this region is in surprisingly good accordance with the diagram in Fig. 7. In our opinion the model presented in Section 4 despite its artificiality, has practical value in the following sense:

(i) it gives an approximation for the lower bound of deflections (outside the region where slots appear);

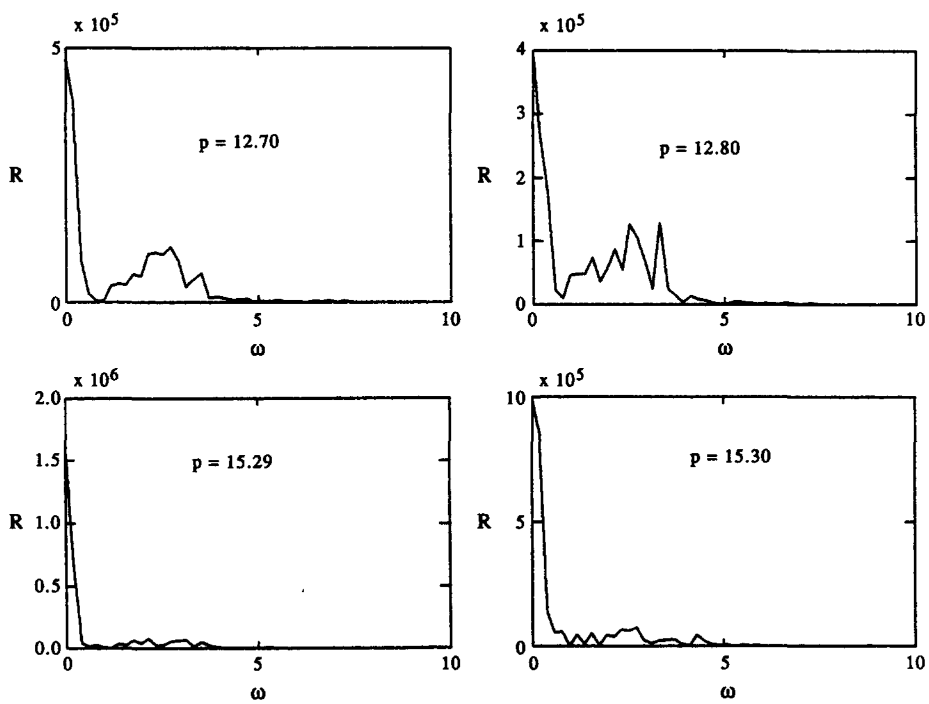

Fig. 10. Power spectra of a beam under uniform transverse loading.

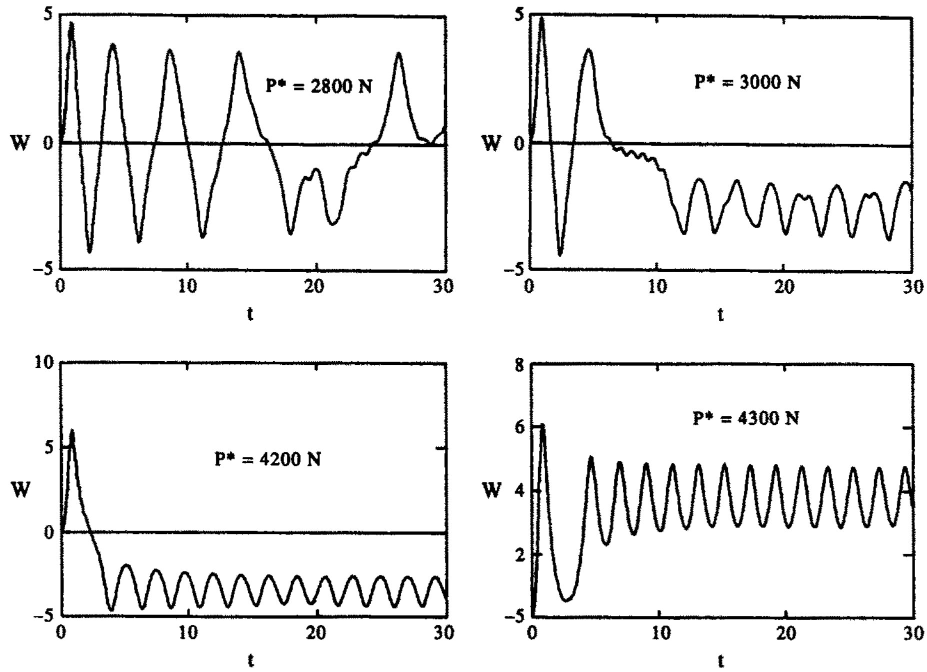

Fig. 11. Mean deflection histories for a beam loaded by force at the midpoint.

(ii) it allows one to calculate algebraically the critical deflection f_m^+ (or critical load p^+), for which the lower bound for the deflection turns positive.

Finally we shall compare briefly our results with the results of Symonds *et al.* for SDoF and 2DoF Shanley beams and with the model of Yu and Xu (1989). As shown in the last mentioned paper, the Shanley type SDoF model and the double-tier-spring model of Yu and Xu are equivalent. Our parameter κ and Yu and Xu's parameter β are connected by the equation

$$\kappa = 2\beta^2. \tag{21}$$

In the case of the diagram in Fig. 7 we have $\kappa = 0.0245$; this corresponds to the value of the Yu and Xu's parameter $\beta = 0.111$. It follows from Fig. 7(a) of the paper by Yu and Xu that only a wide slot, where we have anomalous behavior, exists, but really it follows from our diagram Fig. 7 that there is also a narrow zone of irregular slots. It should be mentioned that the model of Yu and Xu in some cases also predicts two slots (cf. Fig. 7), but the wide slot lies to the left of the narrow slot; the computations by FEM programs [see e.g. Fig. 1 in the paper by Lee *et al.* (1992)] and also our diagram Fig. 7 show that the sequence of these slots is just converse.

As to the 2DoF model of Symonds *et al.* there is not much numerical data available for carrying out more a detailed comparison. Apposition of Fig. 11 and the diagrams in Fig. 9 of the paper by Lee and Symonds (1992) does not give much information (our diagrams seem to be smoother, but this may be due to the fact that we have used the mean deflection instead the midpoint deflection). This problem evidently demands a more detailed investigation.

6. CONCLUSION

The dynamic response of a uniform beam with fixed pin-end constraints under short pulse loading is discussed. This problem is solved by Galerkin's method (with six degrees of freedom). The method of solution proposed by the author demands less computation time than the traditional FEM methods, but in spite of this the accordance of results obtained by both methods is rather good.

The computations, which were carried out for several beam and material parameters, show that weak chaotic effects in the response of the beam may exist, especially in the initial phase; as to the long-term motion then the motion changes to periodic vibrations of smaller amplitude.

An approximate "elastic recovery" type solution is also proposed; this enables us to estimate the lower bound of deflections in the long-term motion. Despite its simplicity this method gives quite satisfactory estimates for minimal deflections; its shortcoming is that it does not describe the slots where the final vibration lies on the negative side of the beam.

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